Convex Relaxations for Manifold-Valued Markov Random Fields With Approximation Guarantees

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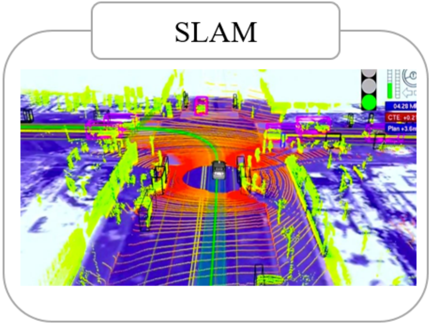
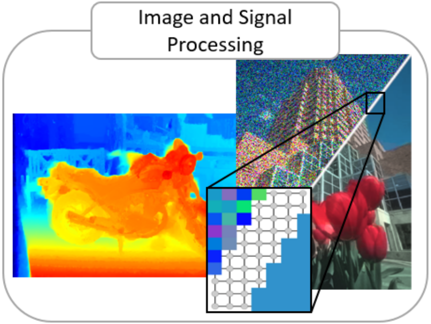
MAP-MRF

$$\underset{x \in \Omega^{\mathcal{V}}}{\text{minimize}} \left\{ F(x) = \sum_{u \in \mathcal{V}} f_u(x_u) + \sum_{(u,v) \in \mathcal{E}} f_{(u,v)}(x_u, x_v) \right\}$$





Maximum A Posteriori Inference for Markov Random Fields

MAP-MRF Problems are Nonconvex and Large-Scale

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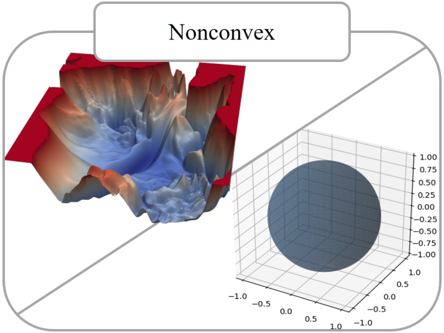

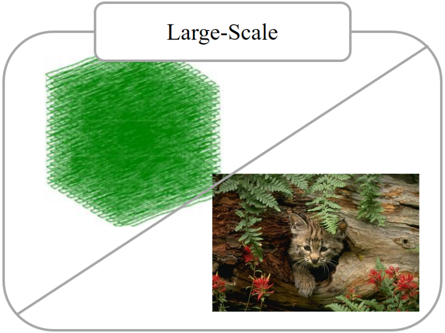

Maximum A Posteriori Inference for Markov Random Fields

The Local Marginal Polytope Relaxation is the Key to Scalable Convex Lifting

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Global Marginal Polytope (GMP)

$$\min_{\mu \in \mathcal{P}(\Omega^{\mathcal{V}})} \langle F, \mu \rangle \quad = \quad \min_{x \in \Omega^{\mathcal{V}}} F(x)$$

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$$\text{IV}$$

Local Marginal Polytope (LMP)

$$\min_{\mu \in \mathcal{P}(\Omega)^{\mathcal{V}}} \sum_{u \in \mathcal{V}} \langle f_u, \mu_u \rangle + \sum_{(u,v) \in \mathcal{E}} \mathrm{OT}_{f_{(u,v)}}(\mu_u, \mu_v)$$

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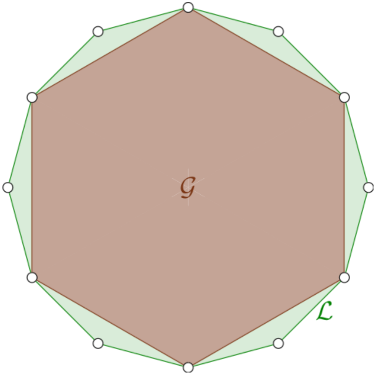

From Optimal Transport Theory to Polynomial Optimization

We rewrite the relaxation as an optimization problem over continuous functions

Local Marginal Polytope - Geodesic Penalty

$$\min_{\mu \in \mathcal{P}(\Omega)^{\mathcal{V}}} \left\{ \quad \mathcal{P}(\mu) = \sum_{u \in \mathcal{V}} \langle f_u, \mu_u \rangle + \sum_{(u,v) \in \mathcal{E}} \mathcal{W}_1(\mu_u, \mu_v) \quad \right\}$$

Kantorovich-Rubinstein Theorem

$$\mathcal{W}_1(\mu_u, \mu_v) = \max_{\phi_{(u,v)} \in \text{lip}(\Omega)} \langle \phi_{(u,v)}, \mu_v - \mu_u \rangle$$

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$$\max_{\phi \in \mathrm{lip}(\Omega)^{\mathcal{E}}} \left\{ \quad \mathcal{D}(\phi) = -\sum_{u \in \mathcal{V}} \sigma_{\mathcal{P}(\Omega)} \left(\nabla_{\mathcal{G}} \cdot \phi - f_u \right) \quad \right\}$$

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$$\text{subject to} \quad \varphi \in \mathrm{lip}_d(\Omega)^{\mathcal{E}}$$

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✓ Finite Dimensional

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- ✓ Increasingly tight

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- ✓ Finite Dimensional
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- ✗ No convergence guarantee

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Convergence:

Let $\Omega = \mathcal{S}^m$, then

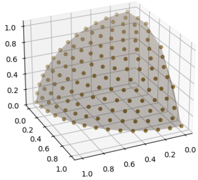
$$\lim_{n \to \infty} \max_{\mathbb{R}[x]_n} \mathcal{D}(\phi) = \max_{\phi \in C(\Omega)} \mathcal{D}(\phi) \qquad \text{subject to} \quad \phi \in \mathrm{lip}(\Omega)^{\mathcal{E}}.$$

Results Using the Semidefinite Relaxation

Our Convex Relaxation may yield Globally Optimal Results even for Low Stages of the Hierarchy

Geodesic TV denoising – Baseline

- Baseline: discretize Ω and solve the discrete MAP-MRF problem using specialized solvers [1, 2]



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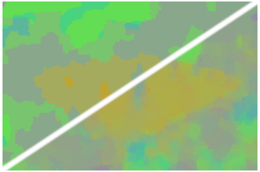
- Baseline suffers from labeling bias



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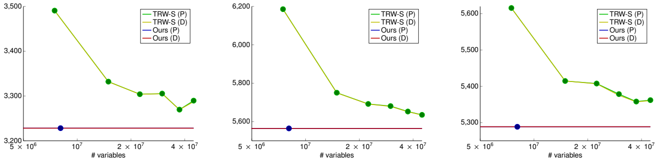

Geodesic TV Denoising – Timing Results

Time = 1000 s		Ours [2]	TRW-S [150]	TRW-S [200]	LR-Solver [150]	LR-Solver [200]
Exp. 2	P	3228.25	3288.05	3266.50	3287.44	3402.78
	D	3228.17	3287.37	3256.48	3287.44	2782.90
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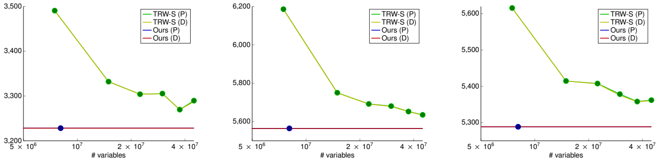

✓ Our method finds the global optimum even in the degree 2 case.

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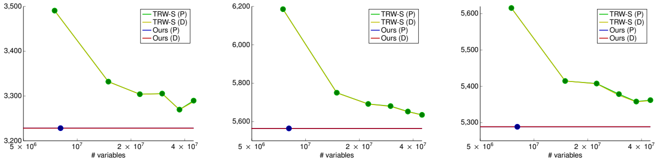

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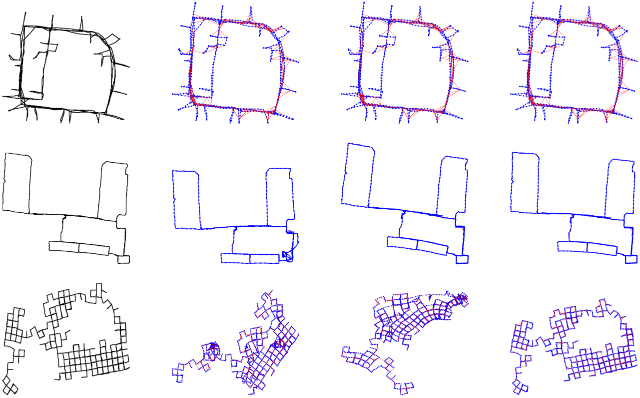
✓ Given a fixed time budget of 1000 seconds, our method significantly outperforms the baseline approach.

Results Using the Semidefinite Relaxation

Our Convex Relaxation Can Be Used to Obtain Good Initializers for Local Methods

Robotics – Pose Graph Optimization



(a) Ground truth (b) Local (c) Ours (d) Ours+local

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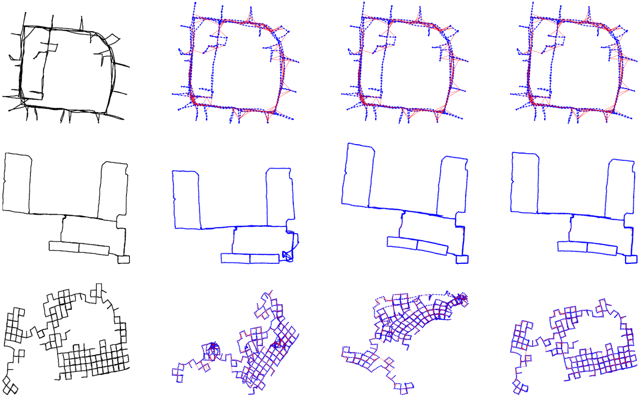

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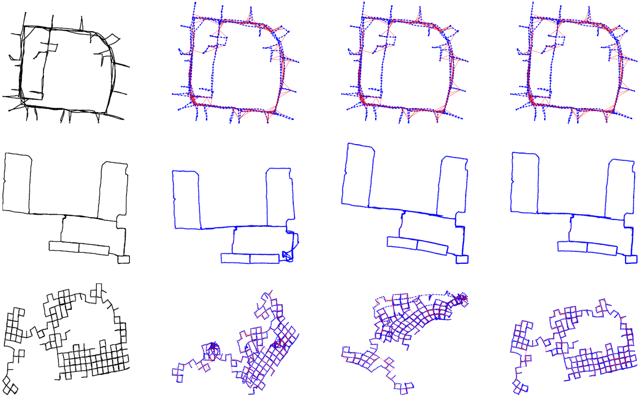

(a) Ground truth (b) Local (c) Ours (d) Ours+local

- ✓ Easy: Near optimal results without rounding
- ✓ General: Optimal results after rounding

Thank you!